Multi-Stage Investment Decision Rules for Power Systems: Sensitivities, Deterministic Equivalents, and Performance Guarantees

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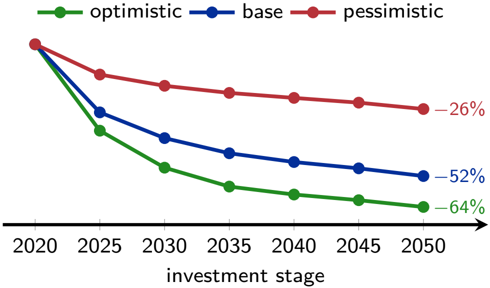
Challenges of the long-term power system planning

- Long-term power system planning is subject to planning uncertainty
- Offshore wind CAPEX from the NREL annual technology baseline:



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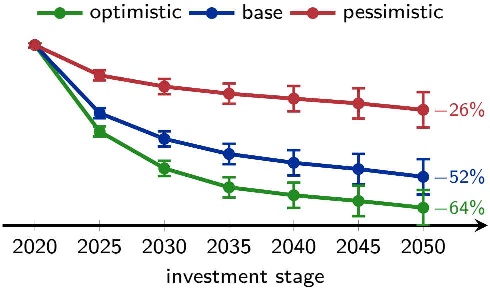

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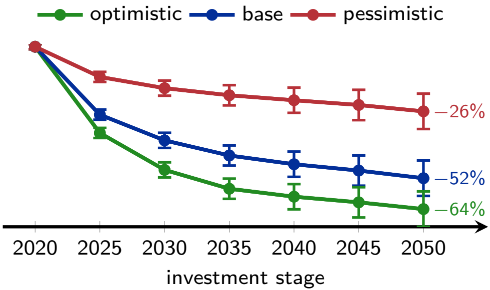

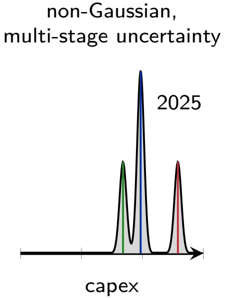

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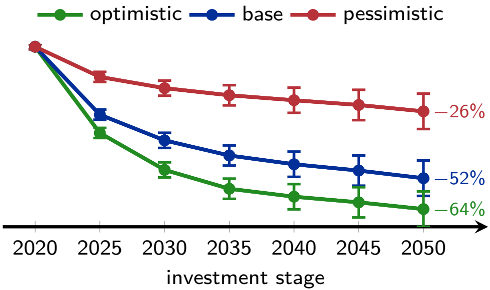

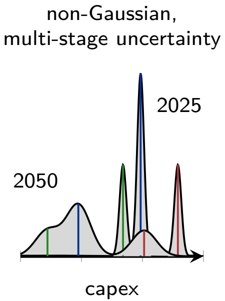

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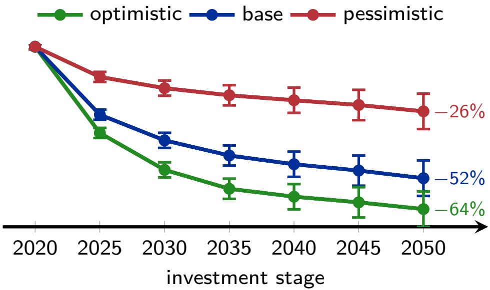
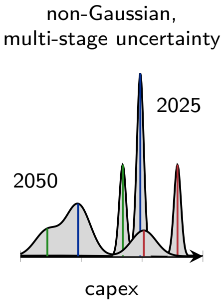
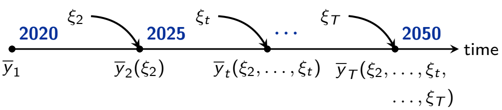




- Planning uncertainty accumulates throughout the investment horizon:



- Scenario # grows exponentially → optimizing dynamic investment decisions $(\overline{y}_1, \ldots, \overline{y}_T)$ becomes intractable even under decomposition (SDDP, progressive hedging. etc.)

Multi-stage linear decision rule (LDR) approximation

$$\begin{aligned}
\min_{y_t,\overline{y}_t} \quad & \mathbb{E}\Big[\textstyle\sum_{t=1}^{T} \Big(\underbrace{\tilde{q}_t(\xi^t)^\top \overline{y}_t(\xi^t)}_{\text{inv. cost}} + \underbrace{\tilde{c}_t(\xi^t)^\top y_t(\xi^t)}_{\text{oper. cost}} \Big) \Big] \\
\text{s.to} \quad & h^{\text{bal}}(y_t(\xi^t)) = \tilde{\ell}_t(\xi^t), \\
& g^{\text{eng}}(y_t(\xi^t), \{\overline{y}_\tau(\xi^\tau)\}_{\tau=1}^{t}) \leqslant 0, \\
& g^{\text{inv}}(\overline{y}_t(\xi^t)) \leqslant 0, \\
& \xi^t \sim \mathbb{P}_{\xi^t}, \quad \forall t = 1, \ldots, T
\end{aligned}$$

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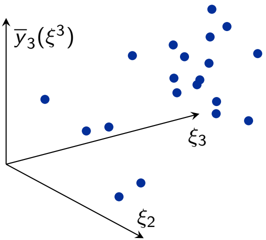
- SAA



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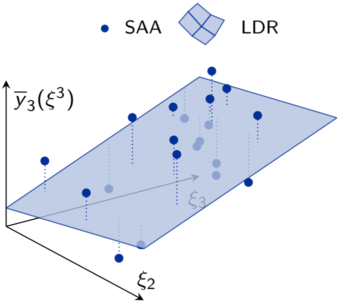

- Linear decision rule approximation [KWG11]:

$$\overline{y}_t(\xi^t) = \overline{Y}_t\xi^t \quad y_t(\xi^t) = Y_t\xi^t$$

with matrices $\overline{Y}_t$ and Y_t to be optimized

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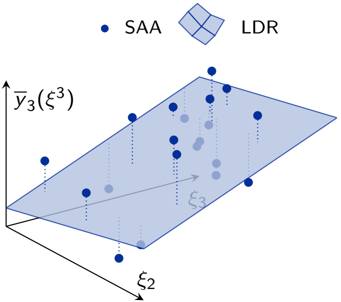

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- How good is the LDR approximation? **This work**:
  1. Providing feasibility guarantees
  2. Optimizing sensitivity to uncertainty
  3. Providing sub-optimality bounds
  4. Learning worst-case scenarios

LDR approximation: Feasibility guarantees

- Power balance (equality constraint) at time stage t and operating hour h:

$$\underbrace{\mathbb{1}^\top \left(\underbrace{Y_{th}\xi^t}_{\text{gen}} - \underbrace{k_h^\ell \circ L_t \xi^t}_{\text{load}} \right) = 0}_{\text{1 equation}} \iff \underbrace{\mathbb{1}^\top \left(Y_{th} - k_h^\ell \circ L_t \right) = 0}_{|\xi^t| \text{ equations}}$$

LDR approximation: Feasibility guarantees

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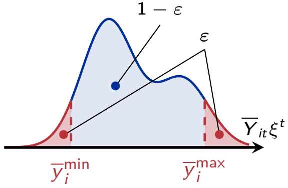
- Limits on investment decision (inequality constraint) in gen. unit i at time stage t:

$$\mathbb{P}_{\xi^t}\left[\overline{y}_i^{\min} \leqslant \overline{Y}_{it}\xi^t \leqslant \overline{y}_i^{\max}\right] \geqslant 1 - \varepsilon$$



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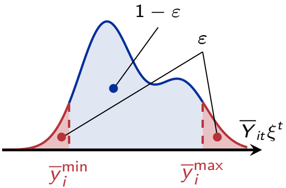
$$\Big\Downarrow \quad \xi^t \sim \mathbb{P}(\mu^t, F^t F^{t\top})$$

scenario-free dist. robust reformulation [XA17] :

$$\sqrt[-2]{\varepsilon}\,\Bigg\|\underbrace{F^t[\overline{Y}_t]_i^\top}_{\delta_{ti}}\Bigg\| \leqslant \tfrac{1}{2}(\overline{y}_i^{\max} - \overline{y}_i^{\min}) - \gamma_{ti}$$

$$\Big|[\overline{Y}_t]_i\mu^t - \tfrac{1}{2}(\overline{y}_i^{\max} - \overline{y}_i^{\min})\Big| \leqslant \delta_{ti} + \gamma_{ti}$$

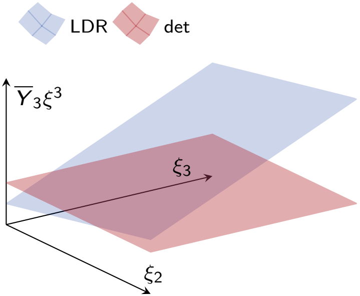
$$\tfrac{1}{2}(\overline{y}_i^{\max} - \overline{y}_i^{\min}) \geqslant \gamma_{ti} \geqslant 0,\ \delta_{ti} \geqslant 0$$

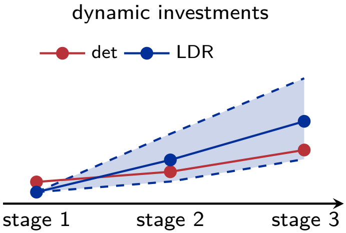


LDR approximation: Investment (in)sensitivity to uncertainty

$$\min_{y_t, \overline{y}_t} \quad \mathbb{E}\Big[\sum_{t=1}^{T} \Big(\tilde{q}_t(\xi^t)^\top \overline{Y}_t \xi^t + \tilde{c}_t(\xi^t)^\top Y_t \xi^t \Big) \Big]$$

$$\begin{aligned} \text{s.to} \quad & h^{\text{bal}}(Y_t \xi^t) = \tilde{\ell}_t(\xi^t), \\ & g^{\text{eng}}(Y_t \xi^t, \{\overline{Y}_\tau \xi^\tau\}_{\tau=1}^{t}) \leqslant 0, \\ & g^{\text{inv}}(\overline{Y}_t \xi^t) \leqslant 0, \\ & \xi^t \sim \mathbb{P}_{\xi^t}, \quad \forall t = 1, \dots, T \end{aligned}$$





- Linear decision rules evaluate the sensitivity of investments to uncertainty

LDR approximation: Investment (in)sensitivity to uncertainty

$$\min_{y_t, \overline{y}_t} \quad \mathbb{E}\Big[\sum_{t=1}^{T}\Big(\tilde{q}_t(\xi^t)^\top \overline{Y}_t\xi^t + \tilde{c}_t(\xi^t)^\top Y_t\xi^t\Big)\Big] + \alpha \mathrm{Var}\Big[\sum_{t=1}^{T} Y_t\xi^t\Big]$$

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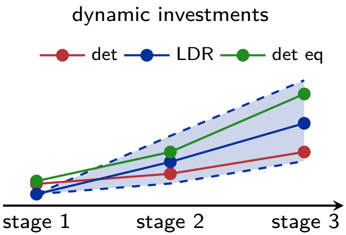
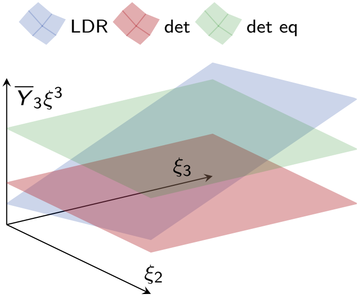
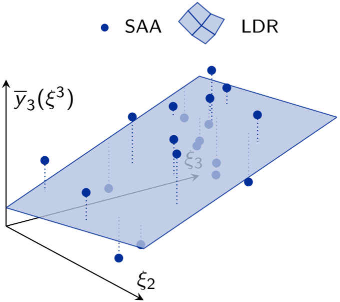




- Linear decision rules evaluate the sensitivity of investments to uncertainty
- The sensitivity can be optimized to meet a trade-off between the expected cost and investment variance (up to penalty α)
- Deterministic equivalent - *insensitive* to uncertainty but *robust* to its realizations

LDR approximation: Sub-optimality bound



- LDR approximation can be sub-optimal
- **$59 billion**: investment into renewable energy in US in 2019
- Even **1%** sub-optimality gap results in the annual loss of **$590 mil.**
- Duality theory to bound sub-optimality [KWG11]

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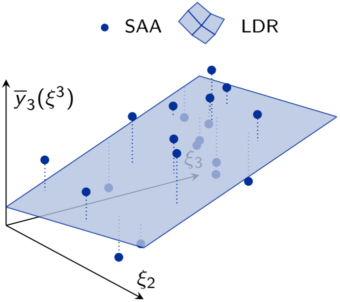

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Primal problem

$$\min_{\overline{y}(\xi)} \quad \mathbb{E}\left[\tilde{c}(\xi)^\top \overline{y}(\xi)\right] \quad \text{s.to} \quad A\overline{y}(\xi) \geqslant \tilde{b}(\xi)$$

strong duality

Dual problem

$$\max_{\lambda(\xi)} \quad \mathbb{E}\left[\tilde{b}(\xi)^\top \lambda(\xi)\right] \quad \text{s.to} \quad A^\top \lambda(\xi) \leqslant \tilde{c}(\xi)$$

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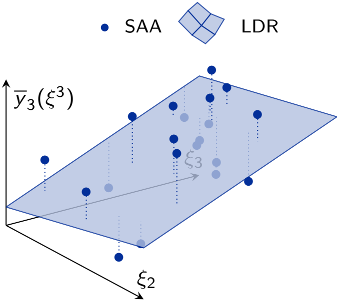

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Primal: $\overline{y}(\xi) = Y\xi$ ↓

Dual: $\lambda(\xi) = \Lambda\xi$ ↓

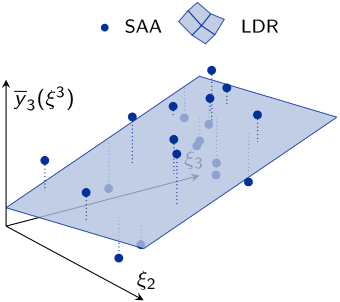
Primal approximation

$$\min_{\overline{Y}} \ \mathbb{E}\left[\tilde{c}(\xi)^\top \overline{Y}\xi\right] \quad \text{s.to} \quad A\overline{Y}\xi \geqslant \tilde{b}(\xi)$$

Dual approximation

$$\max_{\Lambda} \ \mathbb{E}\left[\tilde{b}(\xi)^\top \Lambda\xi\right] \quad \text{s.to} \quad A^\top \Lambda\xi \leqslant \tilde{c}(\xi)$$

LDR approximation: Sub-optimality bound



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Primal problem

$$\min_{\overline{y}(\xi)} \quad \mathbb{E}\left[\tilde{c}(\xi)^\top \overline{y}(\xi)\right] \qquad \text{s.to} \quad A\overline{y}(\xi) \geqslant \tilde{b}(\xi)$$

strong duality

Dual problem

$$\max_{\lambda(\xi)} \quad \mathbb{E}\left[\tilde{b}(\xi)^\top \lambda(\xi)\right] \qquad \text{s.to} \quad A^\top \lambda(\xi) \leqslant \tilde{c}(\xi)$$

Primal: $\overline{y}(\xi) = Y\xi$; Dual: $\lambda(\xi) = \Lambda\xi$

Primal approximation

$$\min_{\overline{Y}} \quad \mathbb{E}\left[\tilde{c}(\xi)^\top \overline{Y}\xi\right] \qquad \text{s.to} \quad A\overline{Y}\xi \geqslant \tilde{b}(\xi)$$

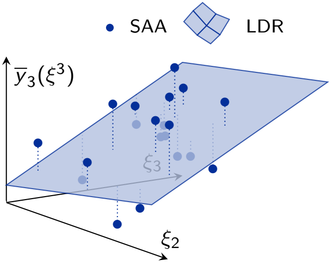
$\neq 0$ duality gap

sub-optimality bound

Dual approximation

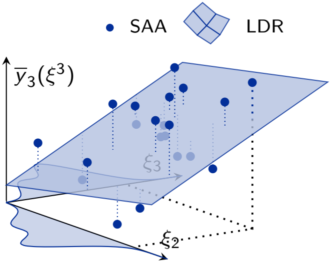
$$\max_{\Lambda} \quad \mathbb{E}\left[\tilde{b}(\xi)^\top \Lambda\xi\right] \qquad \text{s.to} \quad A^\top \Lambda\xi \leqslant \tilde{c}(\xi)$$

LDR approximation: Learning worst-case scenarios



- Duality yields a conservative bound
- What is a likelihood of LDR sub-optimality?
- With small problem instances, we learn the worst-case sub-optimality scenarios

LDR approximation: Learning worst-case scenarios



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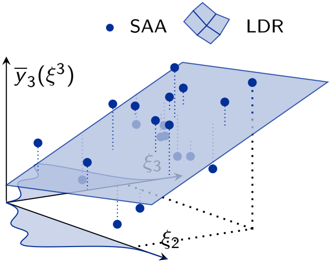

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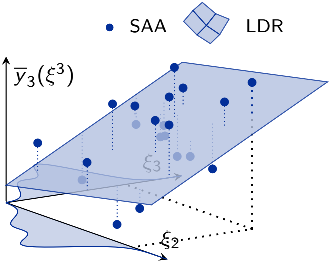
- With small problem instances, we learn the worst-case sub-optimality scenarios

Bilevel optimization-based learning

$$\begin{aligned}
\max_{\xi} \quad & \Big\| \underbrace{c^\top \overline{Y}^\star \xi}_{\text{LDR}} - \underbrace{c^\top (\overline{y}_1^\star + \overline{y}_2)}_{\text{SAA}} \Big\| \\
\text{s.to} \quad & \underline{\xi} \leqslant \xi \leqslant \overline{\xi} \\
& \overline{y}_2 \in \underset{\overline{y}_2}{\text{argmin}} \quad c^\top (\overline{y}_1^\star + \overline{y}_2) \\
& \qquad\qquad \text{s.to} \quad A\,(\overline{y}_1^\star + \overline{y}_2) \geqslant \tilde{b}(\xi)
\end{aligned}$$

- Acts on the support of $\mathbb{P}_\xi$
- Recast as a mixed-integer linear program
- Only right-hand side uncertainty $\tilde{b}(\xi)$

LDR approximation: Learning worst-case scenarios



Sample-based learning

- Duality yields a conservative bound
- What is a likelihood of LDR sub-optimality?
- With small problem instances, we learn the worst-case sub-optimality scenarios

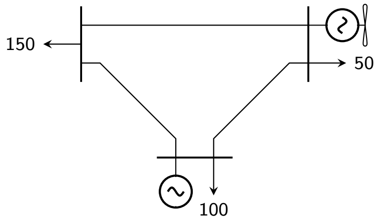
$$\max_{\gamma} \quad \gamma$$
$$\text{s.to} \quad \Big\| \underbrace{\tilde{c}(\xi_s)^\top \overline{Y}^\star \xi_s}_{\text{LDR}} - \underbrace{\tilde{c}(\xi_s)^\top (\overline{y}_1^\star + \overline{y}_{2s}^\star)}_{\text{SAA}} \Big\| \leqslant \gamma$$
$$\forall s = 1, \dots, S$$

where S is the number of samples from

- Acts on samples from $\mathbb{P}_\xi$ [MGL14]
- Recast as a linear program
- Uncertainty of $\tilde{c}(\xi)$ and $\tilde{b}(\xi)$

Illustrative example: System and uncertainty data

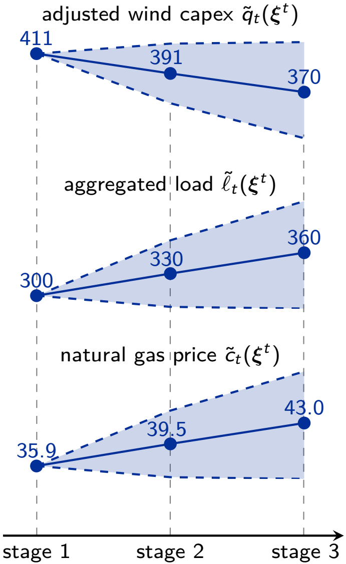
3-bus power system



Tech.	capex	opex
Wind	\$1000/kW	\$0/MWh
CCGT	\$4500/kW	\$35.9/MWh

- Zero initial capacity
- Wind – CCGT competition
- 3-stage investment horizon
- 24 representing operating hours

Uncertainty of planning data



Illustrative example: Deterministic vs. stochastic solutions

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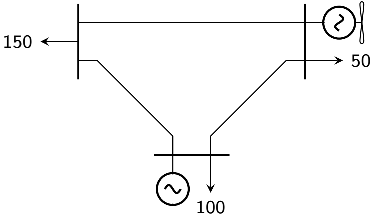
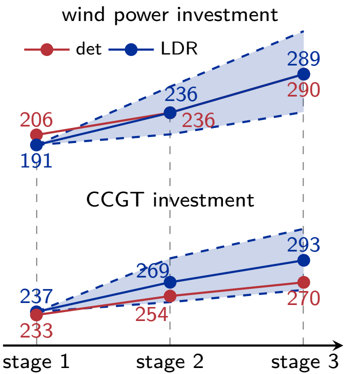


Investment results

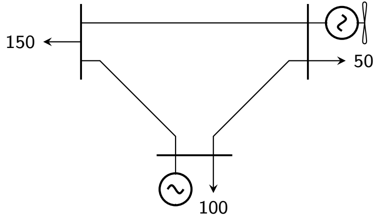


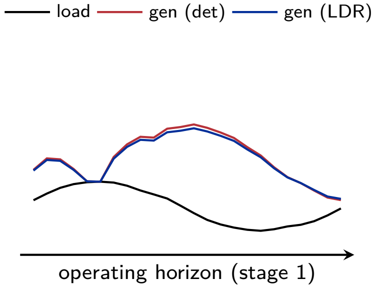
Value of the LDR solution:

- $\left\|\overline{y}_1^{\mathrm{LDR}} - \overline{y}_1^{\mathrm{det}}\right\|_1 \approx$ 20MW or 5%

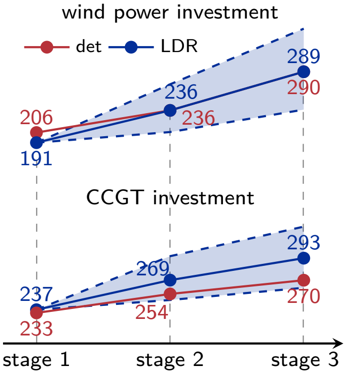
Illustrative example: Deterministic vs. stochastic solutions

3-bus power system





Investment results

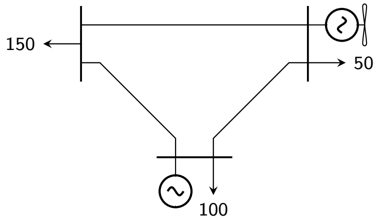


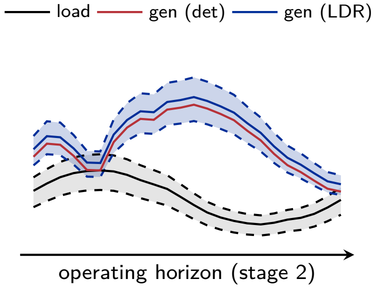
Value of the LDR solution:

- $\left\|\overline{y}_1^{\text{LDR}} - \overline{y}_1^{\text{det}}\right\|_1 \approx$ 20MW or 5%

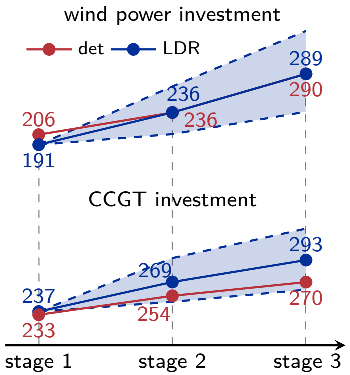
Illustrative example: Deterministic vs. stochastic solutions

3-bus power system





Investment results



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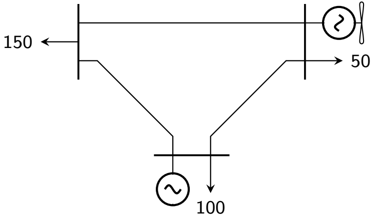
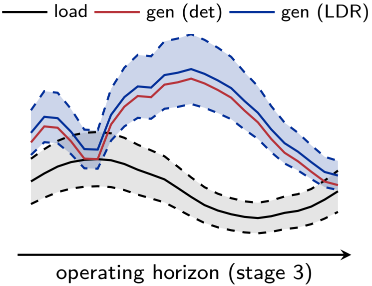
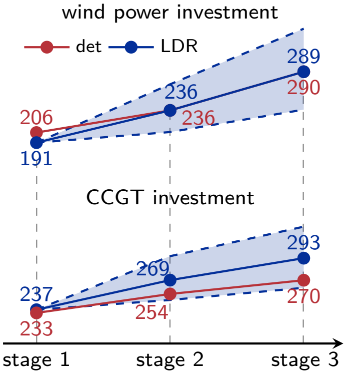




Investment results



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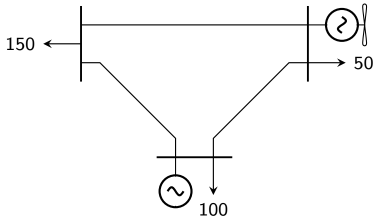
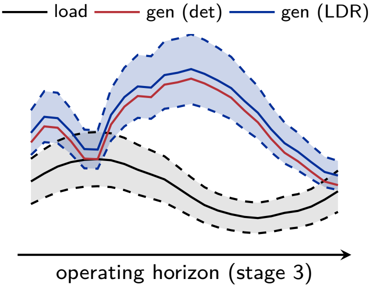
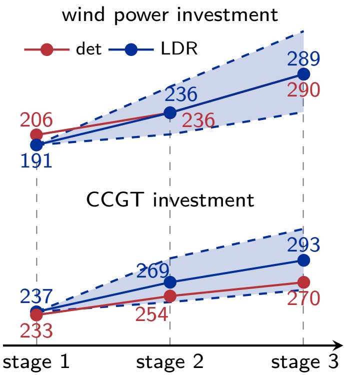




Investment results

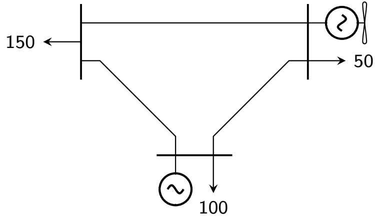


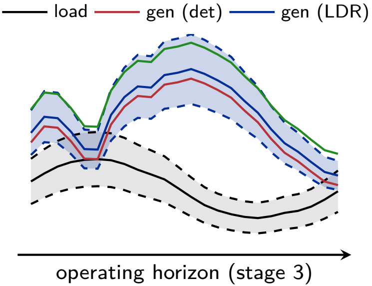
Value of the LDR solution:

- $\left\|\overline{y}_1^{\text{LDR}} - \overline{y}_1^{\text{det}}\right\|_1 \approx$ 20MW or 5%
- For VoLL of \$2,000/MWh, the total cost is:
  - \$713,651 of deterministic solution
  - \$663,568 of stochastic solution

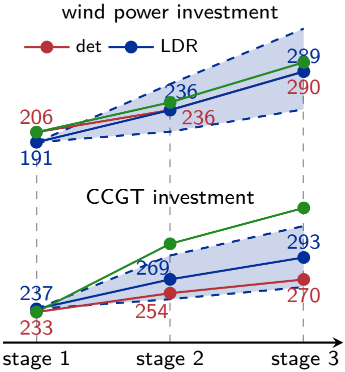
Illustrative example: Deterministic vs. stochastic solutions

3-bus power system





Investment results



Value of the LDR solution:

- $\left|\left|\overline{y}_1^{\text{LDR}} - \overline{y}_1^{\text{det}}\right|\right|_1 \approx$ 20MW or 5%
- For VoLL of \$2,000/MWh, the total cost is:
  - \$713,651 of deterministic solution
  - \$663,568 of stochastic solution
  - \$669,180 of deterministic equivalent

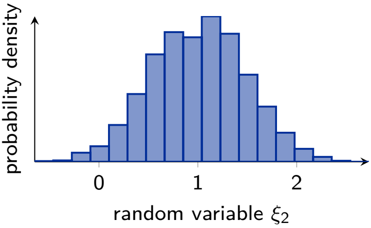
Illustrative example: Optimality loss

- Two-stage uncertainty case
- Expected cost under three approximations:

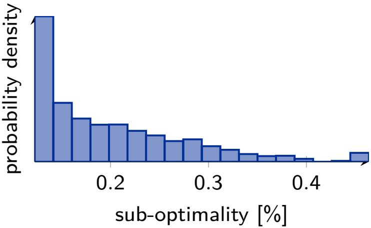
LDR-primal	SAA	LDR-dual
\$412,786	\$412,114	\$410,658

- Duality-based sub-optimality bound: 1%
- Optimization-based learning is sensitive to support bounds ⇒ sample-based learning
- The worst-case sup-optimality is in the left tail of uncertainty distribution:
  - maximal wind capex
  - maximal CCGT opex
  - minimal demand growth

Second-stage uncertainty



Distribution of optimality loss



Illustrative example: Optimality loss

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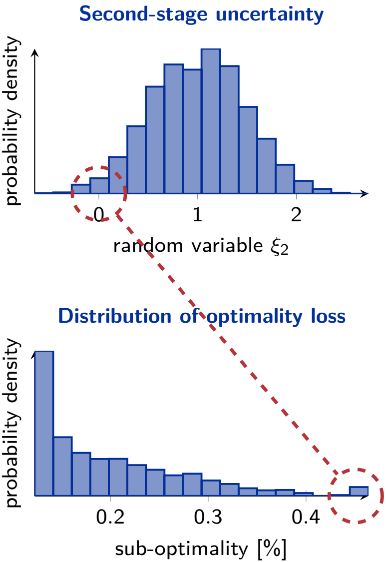
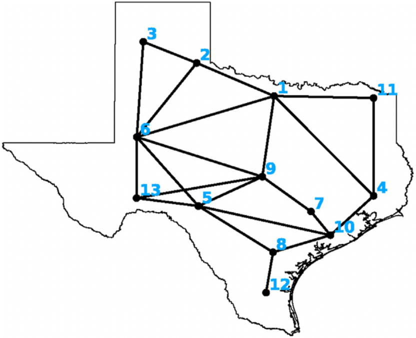
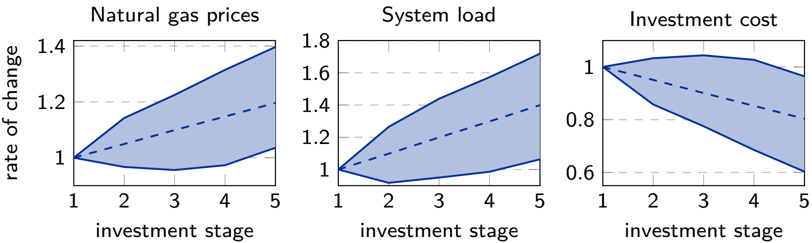


Electric Reliability Council of Texas



- 13 zones, 20 lines, 47 units [BMBK20]
- NREL ATB technology baseline
- 5-stage planning horizon
- 24 operating conditions



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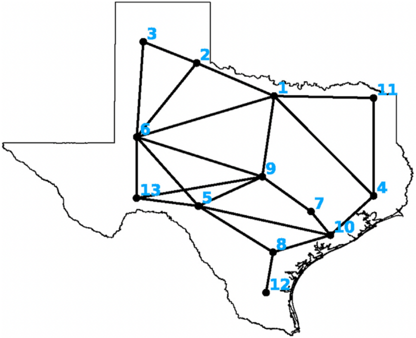

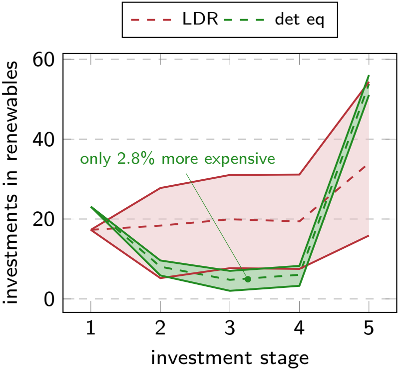

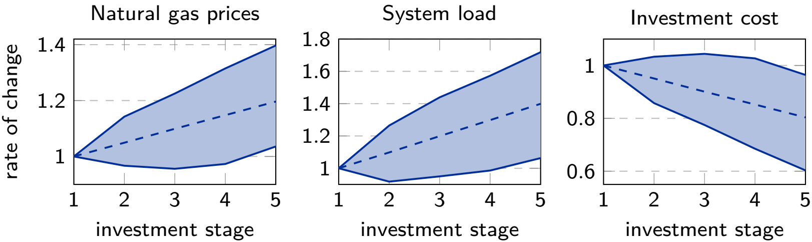

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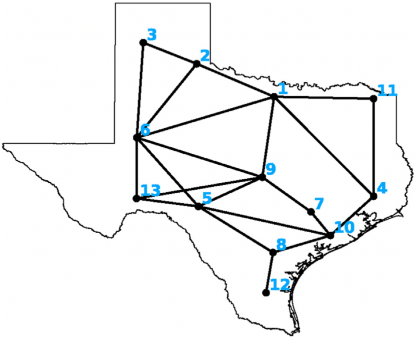

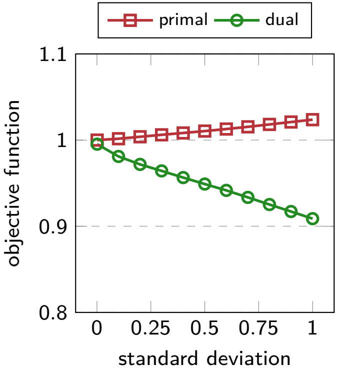

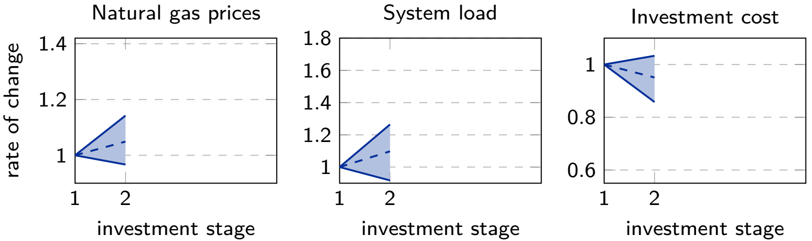

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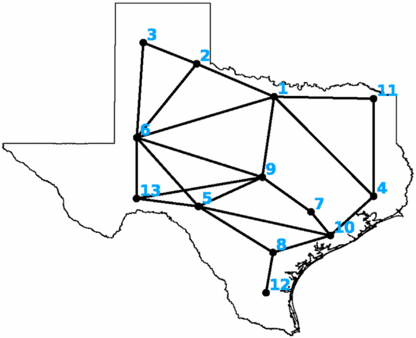

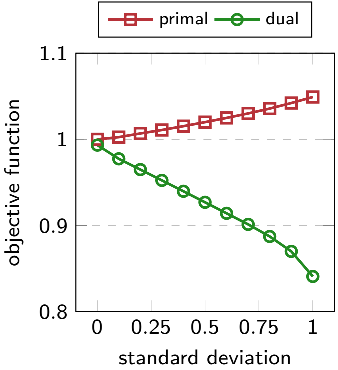

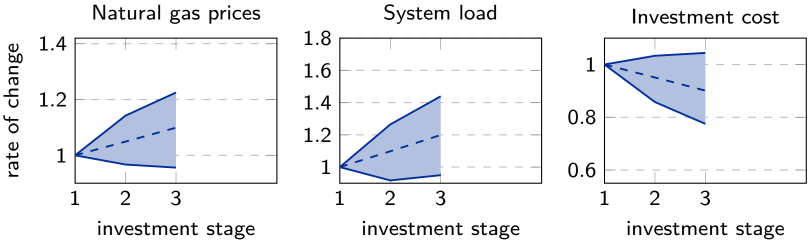

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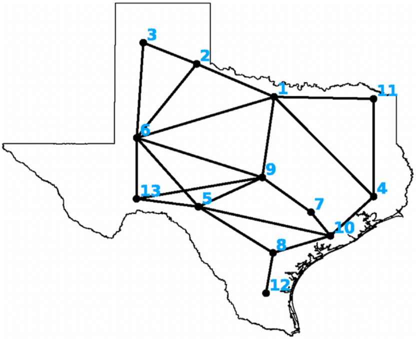

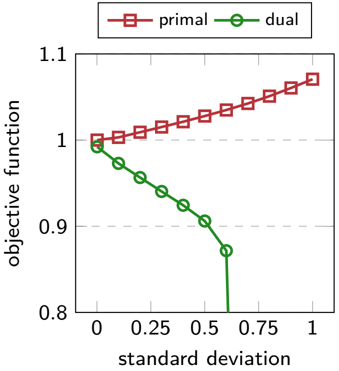

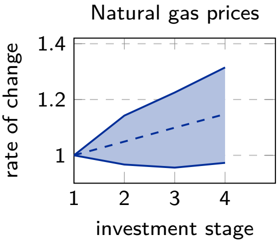

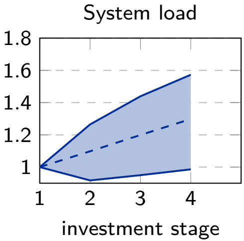

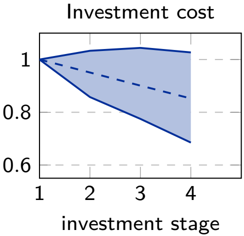

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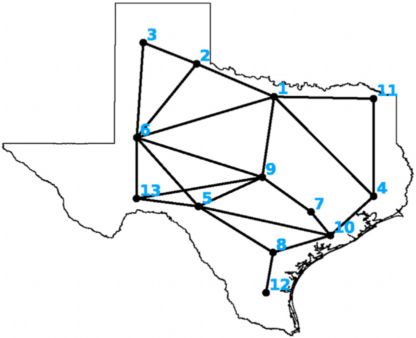
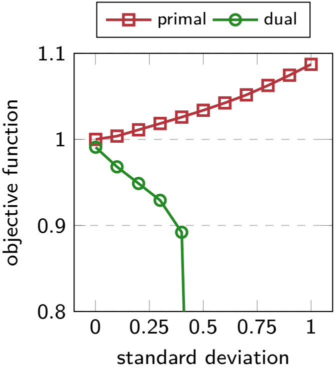
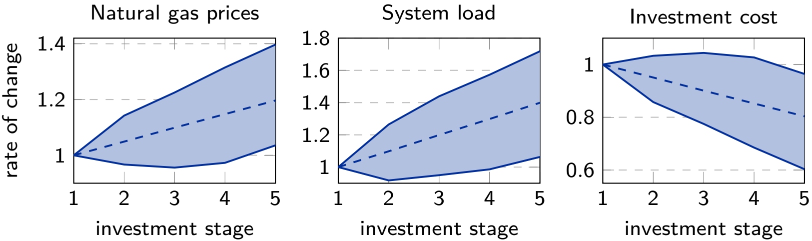
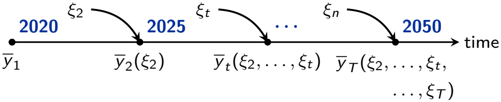






Conclusions

- LDRs approximate the solution of the multi-stage investment problems



- LDRs ensure the feasibility of investment plan under uncertainty
- Sub-optimality of LDRs depends on the magnitude of uncertainty ...
- ... yet the worst-case sub-optimality is at the boundaries of uncertainty set

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